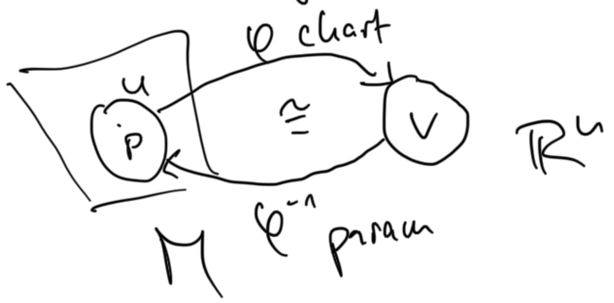
Embedding Manifolds in Euclidean Space

$M^{(n)}$ defined abstractly

Def.: $M^{(n)}$ is n-dim mf. if:

- $M \neq \emptyset$, M is connected, Hausdorff, countable basis
- M is locally Euclidean:



Def: Let $f: M \to N$ diffable. f is called

- immersive in $x \in M \Leftrightarrow d_x f: T_x M \to T_{f(x)} N$ inj.
- immersion $\Leftrightarrow$ everywhere immersive
- f immersion and $f(M) \underset{f}{\cong} M$

Ex.:

immersion

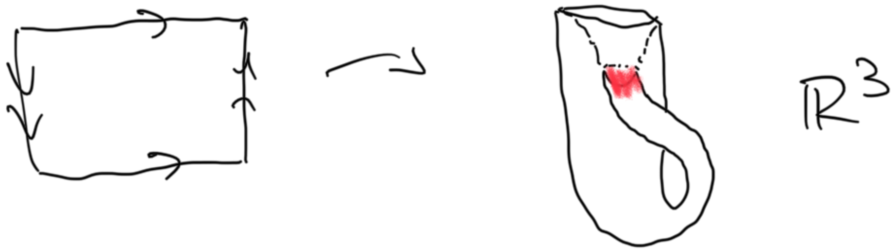
embedding

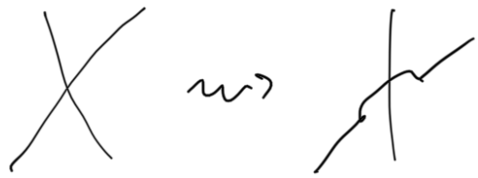
Immersion Embedding

Motivation: Not all n-mf can be embedded in $\mathbb{R}^n$, nor $\mathbb{R}^{n+1}$

Ex: 0) n-spere

1) Klein bottle





Generally: Every closed embedded hypersurfc. in $\mathbb{R}^n$ is orientable

2) $\mathbb{RP}^{2n} \not\hookrightarrow \mathbb{R}^{2n+1}$

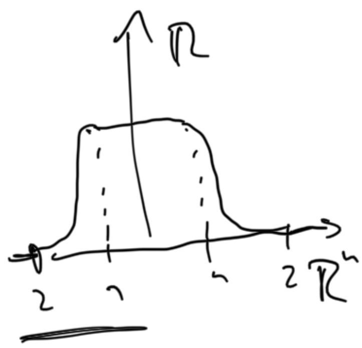
$\uparrow$
not or.

Thm: $M^{(n)}$ compact. Then
(Whitney, version 1) $M^{(n)} \hookrightarrow \mathbb{R}^q$, $q \in \mathbb{N}$

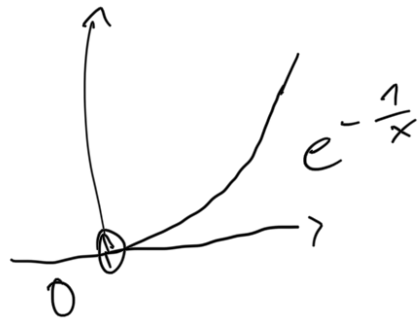
Lemma:

$\exists$ $\lambda : \mathbb{R}^n \to \mathbb{R}$, $\lambda \in C^\infty$ s.t.

$\lambda|_{D(1)} = 1$, $\lambda|_{\mathbb{R} \setminus D(2)} = 0$



Idea:



Lemma:

M, N mflds, $f : M \to N$ inj. immersion

and

a) f proper $f^{-1}(C)$ is compact

compact

01

b) M compact

Then f is an embedding.

Pf (idea):

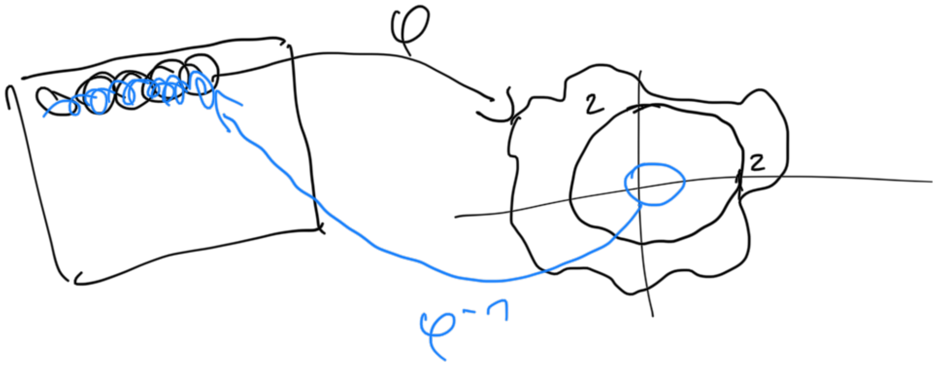
f is closed (Hausdorff).

Proof of WT, version 1:

M is compact $\rightsquigarrow$ finite atlas

$(\varphi_i, U_i)_{i=1}^m \qquad \varphi_i(U_i) \supset D(2)$

$$M = \bigcup_{i=1}^m \varphi_i^{-1}(D(1))$$



Choose $\lambda \in C^\infty(\mathbb{R}^n, \mathbb{R})$

$\underline{III}$

Define $\lambda_i : M \to [0,1]$

$$\lambda_i = \begin{cases} \lambda(\varphi_i(x)) & x \in U_i \\ 0 & \text{else} \end{cases}$$

$$B_i := \lambda_i^{-1}(1)$$

$$f_i(x) = \begin{cases} \underbrace{\lambda_i(x)}_{\mathbb{R}} \cdot \overbrace{\varphi_i(x)}^{\mathbb{R}^n} & x \in U_i \\ 0 & \text{else} \end{cases}$$

$$g_i(x) = \underbrace{(f_i(x), \lambda_i(x))}_{\mathbb{R}^{n+1}}$$

$$g = (g_1, \dots, g_m) : M \to \mathbb{R}^{\underline{(n+1) \cdot m}}$$

Claim: g is an embedding.

IMMERSION:

Choose $x \in M$, $\operatorname{rk} d_x g = \underline{\underline{\operatorname{rk} d_x \varphi_i}}$

where $x \in B_i$

$\Rightarrow$ g is immersive in x

$\Rightarrow$ a immersion.

INJECTIVITY

Let $x, y \in M$, $x \neq y$, $y \in B_j$

- $x \in B_j$

$$g_j(y) \neq g_j(x)$$
$$\shortparallel \qquad\qquad \shortparallel$$
$$(\varphi_j(y), \underline{1}) \neq (\varphi_j(x), \underline{1})$$

- $x \notin B_j \Rightarrow g$ injective

$\overset{\text{Lem.}}{\Longrightarrow}$ g embedding. $\square$

Rmk: m can be huge.

Thm (Whitney, version 2)

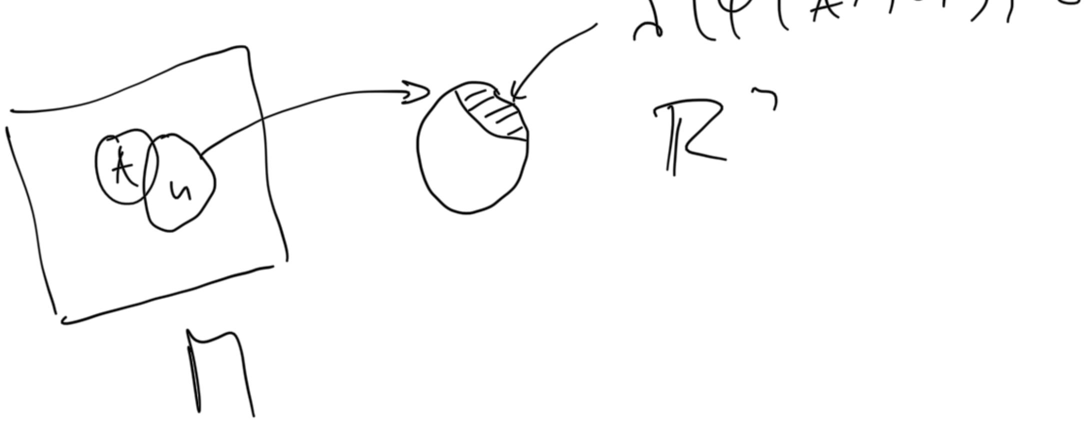
M^n can be immersed injectively in $\mathbb{R}^{2n+1}$.

Lemma (Sard's theorem):

Let $f: N^{(n)} \to M^{(m)}$, diffable $n < m$. Then $f(N)$ has measure 0.

Pf: Pollack

Rmk:



Rmk: "Diffable" is needed.

Def: Tangent bundle:

$$T(M) := \{ (x, v) \in x \in M, v \in T_x M \}$$

Prop: $T(M^{(n)})$ is a $2n$-mf.

Proof: $T(M) \cap (\underset{\text{open in } M}{W} \times \mathbb{R}^n) \underset{d(\varphi^{-1})}{=} \overbrace{\underset{\text{open in } \mathbb{R}^n}{T(U)} \times \mathbb{R}^n}^{2n}$

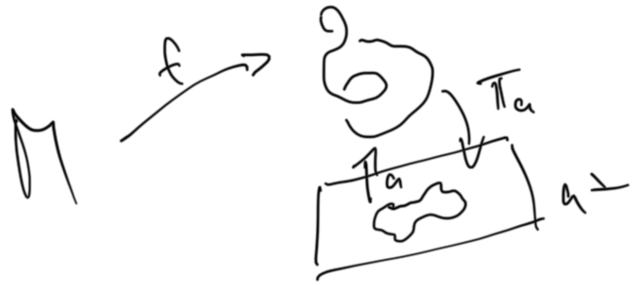
Proof of version 2:

goal: inj. immersion $f: M \to \mathbb{R}^{2n+1}$

By version 1: M inj. immersed in $\mathbb{R}^N$

if $N \leq 2n+1$ $\rightsquigarrow$ done

Assume that $N > 2n+1$



Def.:

$$h : M^n \times M^n \times \mathbb{R}^1 \xrightarrow{<} \mathbb{R}^N$$
$$(x, y, t) \mapsto t(f(x) - f(y))$$
$$g : T(M)^{2n} \xrightarrow{<} \mathbb{R}^n$$
$$(x, v) \mapsto d_x f(v)$$

By Sard's thm:

$$\underbrace{\mathrm{Im}(h) \cup \mathrm{Im}(g)}_{} \neq \mathbb{R}^N$$

Pick $a \in \mathbb{R}^N \setminus$

in part. $a \neq 0$

Claim $\pi_a \circ f$ is inj. immersion.

INJECTIVITY:

- $x, y \in M \quad x \neq y$, assume that

$\pi_a \circ f(x) = \pi_a \circ f(y)$

$\Rightarrow f(x) - f(y) = t a$

$\Rightarrow t \neq 0 \Rightarrow a \in \mathrm{Im}(h)$ ✓

IMMERSION

Let $0 \neq v \in T_x(M)$ s.t. $d_x(\pi_a \circ f)^{(v)} = 0$

$\pi_a \circ \underbrace{d_x(v)}_{t \cdot a} = 0 \qquad d_x f(v) = t \cdot a$

$\Rightarrow a \in \mathrm{Im}(g)$ ↯

$\Rightarrow \pi_a \circ f$ is inj. immersion. □

Corollary (Whitney version 2.1)

If $M^{(n)}$ is compact, it admits an embedding $M \hookrightarrow \mathbb{R}^{2n+1}$

Proof: (last thm + lemma)

Idea: make f proper.

Lemma: $\exists$ proper function $g : M^{(n)} \to \mathbb{R}$

Idea: $\{U_i\}$ open subsets that have comp. closure. P.o.u. $\{\theta_i\}$

$$g := \sum_{i=1}^{\infty} i\, \theta_i$$

Thm (Whitney, version 3)

$M^{(n)} \hookrightarrow \mathbb{R}^{2n+1}$

Pf:

By version 2: $f : M \to \mathbb{R}^{2n+1}$, $|f(x)| < 1$

Let $g : M \to \mathbb{R}$ proper. Define the inj. immersion

$$F : \underline{M} \to \mathbb{R}^{2n+2}$$
$$x \mapsto (f(x), g(x))$$

inj: f inj.

immersion : f immersion

Let $a \in S^{2n+1}$ be a vector s.t.

$\pi_a \circ F$ is inj. immersion and $\underline{a_{2n+2} \in \{\pm 1\}}$

Ts. "proper"

Claim: $\exists\, d \in \mathbb{R}$ s.t. for $\underline{K \text{ compact}}$

$$x \in (\pi_a \circ F)^{-1}(K) \Rightarrow \underline{|\varrho(x)|} \le d$$

if claim is true:

$\varrho((\pi_a \circ F)^{-1}(K))$ is bounded + closed

$\Rightarrow$ comp. $\overset{\varrho \text{ proper}}{\Longrightarrow}$ $(\pi_a \circ F)^{-1}(K)$ compact.

Pf of the claim.

Suppose not: $\exists (x_i) \subseteq M$ s.t.

$$|(\pi_a \circ F)(x_i)| \leq C \quad \text{for } K \subseteq B_c(0)$$

and

$$g(x_i) \xrightarrow[i\to\infty]{} \infty$$

$$w_i := \frac{1}{g(x_i)} \underbrace{\left(F(x_i) - \pi_a \circ F(x_i)\right)}_{\perp a} \quad \perp$$

Consider:

$$\frac{F(x_i)}{g(x_i)} = \left(\underbrace{\frac{f(x_i)}{g(x_i)}}_{|\dots| < 1,\ g \to \infty}, 1 \right) \longrightarrow (0, 0, \dots, 0, 1)$$

$$\underbrace{\frac{1}{g_i}}_{\to 0} \underbrace{\pi_a \circ F(x_i)}_{|\dots| \leq C} \longrightarrow 0$$

$$w_i \longrightarrow (0, \dots, 0, 1)$$

$\uparrow$ mult. of a $\qquad$ $\uparrow$ mult. of a $\qquad$ $\lightning$

$\square$

Corollary:

Corollary:

$$M^{(n)} \hookrightarrow \mathbb{R}^{2n}$$

Thm (Whitney, final version)

$M^{(n)}$ can be embedded in $\mathbb{R}^{2n}$.

Rmk:

$$\mathbb{RP}^{2^k} \not\hookrightarrow \underline{\mathbb{R}^{2\cdot 2^k - 1}}$$

But: $M^{(3)}$ compact $\looparrowright \mathbb{R}^5$

Also true: $M^{(3)} \hookrightarrow \mathbb{R}^5$ (cf. C.T.C. Wall: "All 3-Mfds. Imbed In 5-Space")

$\rightsquigarrow$ Better bound for immersion

$$M^{(n)} \looparrowright \mathbb{R}^{2n - \underline{\alpha(n)}}$$

$\alpha(n) = \#\,1$'s in bin. repr. of n

Cohen 1985

WT in context of approx

Thm:

$g: M^{(n)} \to \mathbb{R}^k$ diffable, $k \geq 2n+1$, $\varepsilon > 0$. $\exists\ f: M \hookrightarrow \mathbb{R}^k$ s.t.

$$|f(x) - g(x)| < \quad \forall x \in M.$$

Idea: any emb. $h : M \to \mathbb{R}^s$

$$\begin{tikzcd}
H = (g,h) : M \arrow[r] \arrow[dr, bend right, "g"] & \mathbb{R}^k \times \mathbb{R}^s \arrow[d, "\pi_k"] \\
& \mathbb{R}^k
\end{tikzcd}$$

$\rightsquigarrow$ Approximation of π_k by f s.t. $f|_M$ stays an emb.